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THE NON-EXISTENCE OF RADIAL SOLUTIONS TO NONLINEAR CHERN-SIMONS-SCHRÖDINGER SYSTEMS WITH SOME SPECIAL NONLINEARITIES

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Abstract

In this paper, we consider the nonlinear Chern-Simons-Schrödinger systems with special nonlinearities and external potentials. We prove that this problem has no nontrivial radial solution when the parameter λ is large enough.

1. Introduction

The Chern-Simons-Schrödinger system (CSS system) was proposed by [3] and [4], which describes the feature of high-temperature super-conductor, quantum Hall effect, and Aharovnov-Bohm scattering etc. By using the ansatz and the Coulomb gauge condition which was mentioned in [1] and [2], the CSS system gives the following problem:

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$$-\Delta u+\omega u+\left(\xi+\int_{|x|}^{\infty}\frac{h(s)}{s}u^{2}(s)ds\right)u+\frac{h^{2}(|x|)}{|x|^{2}}u=g(u)\ \text{in}\ \mathbb{R}^{2}, \tag{1.1}$$

where $h(s)=\int_{0}^{s}\frac{\tau}{2}u^{2}(\tau)d\tau,\ g(u)=\lambda|u|^{p-1}u,$ and $w,\ \xi\in\mathbb{R}.$

The existence, non-existence, and multiplicity of radial standing waves to (1.1) were discussed by [1] and [2], where the authors study that $g(u)$ is super linear.

In this paper, we consider the non-existence of radial solutions to the following Schrödinger equation with the gauge field:

$$-\Delta u+V(x)u+\lambda\left(\int_{|x|}^{\infty}\frac{h(s)}{s}u^{2}(s)ds+\frac{h^{2}(|x|)}{|x|^{2}}\right)u=g(u)\ \text{in}\ \mathbb{R}^{2}, \tag{1.2}$$

where $h(s)=\int_{0}^{s}\frac{\tau}{2}u^{2}(\tau)d\tau,\ \lambda>0,\ V(x)$ and $g(u)$ satisfy the following hypotheses:

(V1) $V(x)\in C(\mathbb{R}^{2},\ \mathbb{R})$ and $V(x)\equiv V(|x|)\geq a>0$ for all $x\in\mathbb{R}^{2}$.

(V2) $\lim_{|x|\to\infty}V(x)=V(\infty)\in(0,\ +\infty)$.

(g1) $g(u)=au^{n}|\sin u|$, where a is defined in (V1) and $n=2,\ 3$.

By variational methods, the authors in [5] obtain the existence and multiplicity of radial solutions to (1.2) depending on the parameter λ when $V(x)$ is a radial symmetric positive function and $g(u)$ is asymptotical linear. They also prove that (1.2) has no nontrivial radial solution for λ large enough.

Inspired by the results we mentioned above, we are interested in the non-existence of radial solutions to (1.2) for $V(x)$ is a radial symmetric positive function and $g(u)=au^{n}|\sin u|$, where $a\in\mathbb{R}^{+}$ and $n=2,\ 3$. For $n=1$, we have $0\leq\frac{g(u)}{u}\leq a,\ \forall u\neq 0$. The authors in [6] obtain that

(1.2) has no nontrivial radial solution for every $\lambda > 0$ when $n = 1$. Obviously, here for $n = 2, 3$, $g(u)$ is neither asymptotical linear nor super linear and $\frac{g(u)}{u}$ is not bounded. By the method of [5], we can obtain the following main result:

Theorem 1.1. *Assume that* $V(x)$ *satisfies* (V1), (V2), *and* (g1) *holds, then there exists* $\lambda^* > 0$ *such that* (1.2) *has no nontrivial radial solution for* $\lambda \geq \lambda^*$.

2. Non-Existence of Radial Solutions

In this section, we prove the non-existence of radial solutions to (1.2), that is, Theorem 1.1. Here we use standard notations. $H_r^1(\mathbb{R}^2)$ denotes a radial Sobolev space equipped with the norm:

$$\|u\| := \left(\int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 dx\right)^{1/2},$$

which is equivalent to

$$\|u\|_V := \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)|u|^2 dx\right)^{1/2}.$$

Let us define the functional $\mathcal{I}_\lambda : H_r^1(\mathbb{R}^2) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx + \frac{\lambda}{2}\mathcal{P}(u) - \int_{\mathbb{R}^2} G(u) dx, \tag{2.1}$$

where $\mathcal{P}(u) := \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{\tau}{2} u^2(\tau) d\tau\right)^2 dx$ and $G(u) = \int_0^u g(s) ds$.

Similar to [1] and [5], we have the following result:

Lemma 2.1. *The functional* $\mathcal{I}_\lambda$ *is continuously differentiable on* $H_r^1(\mathbb{R}^2)$ *and its critical point* u *is a weak solution of* (1.2). *Moreover, a* *critical point* u *of* $\mathcal{I}_\lambda$ *belongs to* $C^2(\mathbb{R}^2)$, *so the weak solution* u *is a classical solution of* (1.2).

Let us recall an inequality in [5], which we will use in our proof of main theorem.

Lemma 2.2. *For* $u \in H_r^1(\mathbb{R}^2)$, *we obtain that for every* $\varepsilon > 0$, *the following inequality holds*:

$$\int_{\mathbb{R}^2} |u|^4 dx \leq 2\varepsilon \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{2}{\varepsilon} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx.$$

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $u \in H_r^1(\mathbb{R}^2)$ is a solution of (1.2). We obtain that there exists $C = C(a) > 0$ such that

$$g(u)u = \begin{cases} au^3|\sin u| \leq au^2 + C(a)u^4, & n = 2, \\ au^4|\sin u| \leq a|u|^4, & n = 3, \end{cases} \leq au^2 + C|u|^4, \tag{2.2}$$

where $C(a) = a$ when $n = 3$.

Multiplying the Equation (1.2) by u and integrating by parts, we get

$$\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx + \lambda \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2} \right) u^2 dx$$
$$- \int_{\mathbb{R}^2} g(u)u dx = 0. \tag{2.3}$$

By (2.3), (2.2), and Lemma 2.2, choosing $\frac{1}{\sqrt{2\lambda}} < \varepsilon < \frac{1}{\sqrt{\lambda}}$, we have

$$0 = \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx + \lambda \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2} \right) u^2 dx$$
$$- \int_{\mathbb{R}^2} g(u)u dx$$

$$\geq \int_{\mathbb{R}^2} |\nabla u|^2 + (V(x) - a)u^2 dx + \lambda \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx - \int_{\mathbb{R}^2} C|u|^4 dx$$

$$\geq (1 - \lambda\varepsilon^2) \int_{\mathbb{R}^2} |\nabla u|^2 dx + \left(\frac{\lambda\varepsilon}{2} - C\right) \int_{\mathbb{R}^2} |u|^4 dx$$

$$\geq \left(\frac{\lambda\varepsilon}{2} - C\right) \int_{\mathbb{R}^2} |u|^4 dx.$$

Hence, if $\lambda > 8C^2$, then u must be zero. □

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